STEADY-STATE ISOTHERMAL FLOW OF A VISCOPLASTIC DISPERSED SYSTEM IN AN ELBOW PIPE OF CIRCULAR CROSS SECTION

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The equation of flow of a viscoplastic system in an elbow pipe of noncircular cross section is presented in natural form. Using the equation so derived, an approximate solution is obtained for the motion of a viscoplastic system in an elbow pipe of circular cross section.

The Shvedov–Bingham method has been employed for studying the flow of many dispersed systems. The chief advantage of this method over other methods lies in the simplicity of the equations. Experimental data show that the Shvedov–Bingham equation is obeyed quite closely for high rates of deformation even by non-Newtonian liquids. Calculations indicate that the neglect of nonlinearity in the region of small deformations (for example, in the paraxial region for motion in a circular cylinder) introduces negligible errors from the point of view of practical engineering, but greatly simplifies the computing method.

Shul'man [2] showed that the Kesson equation gave a fair description of the flow of a pseudoplastic liquid, and proposed a more general equation applicable to systems with complex rheological properties. A number of theoretical problems have now been solved by means of these equations. Nevertheless, the Shvedov–Bingham model, being the simplest representation and giving a fair description of many real materials, has never lost its importance.

Let us consider the flow of a highly viscous Shvedov–Bingham liquid in a circular pipe with its axis bent into the arc of a circle of radius R. End effects on entering and leaving the pipe will be neglected. This is justifiable for highly viscous liquids in which the initial section in which flow starts developing is fairly small.

This problem may find practical application in calculating the motion of such viscoplastic materials as peat, clay, etc. in the profiled parts of machinery.

Let us suppose that r_0 is the radius of cross section of the pipe and α is the central angle of the arc AB (Fig. 1).

We consider that in the cylindrical coordinate system r, θ, z

$$v_r = v_z = 0; \quad v_\theta = \varphi(r, z).$$

The current lines will thus be circular arcs lying in planes parallel to the coordinate plane $ro\theta$. The centers of these circles lie in the oz axis.

The deformation-velocity tensor components will be

$$\dot{e}_{rr} = \dot{e}_{\theta\theta} = \dot{e}_{zz} = 0,$$

$$\dot{e}_{r\theta} = \frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r}, \quad \dot{e}_{\theta z} = \frac{\partial v_\theta}{\partial z}, \quad \dot{e}_{zr} = 0.$$

The intensity of the deformation velocities

$$h = \sqrt{\left(\frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r}\right)^2 + \left(\frac{\partial v_\theta}{\partial z}\right)^2}.$$

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We thus obtain the following differential equations for the rheodynamics of the viscoplastic system:

$$\frac{\partial P}{\partial r} = \rho_0 \frac{v_\theta^2}{r}, \tag{1}$$

$$-\frac{\partial P}{r\partial\theta} + \left(\eta + \frac{\tau_0}{h}\right)\left(\frac{2\dot{e}_{r\theta}}{r} + \frac{\partial \dot{e}_{r\theta}}{\partial r} + \frac{\partial \dot{e}_{\theta r}}{\partial z}\right) - \frac{\tau_0}{h^2}\left(\frac{\partial h}{\partial r}\dot{e}_{r\theta} + \frac{\partial h}{\partial z}\dot{e}_{\theta z}\right) = 0, \tag{2}$$

$$\frac{\partial P}{\partial z} = 0. \tag{3}$$

If we neglect the right-hand side (representing the normal inertial force) in Eq. (1), P may be a function of θ only. However, the deformation-velocity tensor components in Eq. (2) are functions of r and z. Hence P will be a linear function of θ.

Denoting the pressure at the entrance into the pipe by P_1 and that at the exit by P_2, we obtain

$$P = \frac{(P_1 - P_2)}{\alpha}\theta + P_1.$$

The stress components

$$P_{rr} = P_{\theta\theta} = P_{zz} = -P,$$

$$P_{r\theta} = \left(\eta + \frac{\tau_0}{h}\right)\dot{e}_{r\theta},\quad P_{z\theta} = \left(\eta + \frac{\tau_0}{h}\right)\dot{e}_{\theta z},\quad P_{zr} = 0.$$

Denoting

$$\frac{P_1 - P_2}{\alpha} = -i,$$

Eq. (2) may be rewritten in stress form

$$\frac{i + 2P_{r\theta}}{r} + \frac{\partial P_{r\theta}}{\partial r} + \frac{\partial P_{z\theta}}{\partial z} = 0. \tag{4}$$

We now express the velocity v_θ in the form

$$v_\theta = r\Phi(r, z).$$

Then we obtain the following expression for the intensity of the deformation velocities

$$h = r\sqrt{\left(\frac{\partial\Phi}{\partial r}\right)^2 + \left(\frac{\partial\Phi}{\partial z}\right)^2}.$$

We introduce the new variables

$$x = r - R,\quad z = z_1,\quad -r_0 \leqslant x \leqslant r_0,\quad -r_0 \leqslant z_1 \leqslant r_0.$$

Then putting

$$\Phi(r, z) = \nu(x, z_1),$$

we obtain

$$h = r\sqrt{\left(\frac{\partial\nu}{\partial x}\right)^2 + \left(\frac{\partial\nu}{\partial z_1}\right)^2} = \pm r\frac{\partial\nu}{\partial n},$$

where n is the external normal to the isoline

$$\nu(x, z_1) = C.$$

Let us express the following components in terms of $\nu(x, z_1)$

$$P_{r\theta} = \left(\eta r\frac{\partial\nu}{\partial n} \pm \tau_0\right)\frac{\left(\frac{\partial\nu}{\partial x}\right)}{\left(\frac{\partial\nu}{\partial n}\right)}, \tag{5}$$

$$P_{z\theta} = \left(\eta r \frac{\partial v}{\partial n} \pm \tau_0\right) \frac{\left(\frac{\partial v}{\partial z_1}\right)}{\left(\frac{\partial v}{\partial n}\right)}. \tag{6}$$

In the brackets the sign in front of τ_0 has to be chosen so that it may coincide with the sign of the normal derivative $\partial \nu/\partial n$.

Essentially the expression $\pm\tau_0 + \eta r(\partial \nu/\partial n)$ represents the tangential stress at the slip surface. Denoting this by τ, we differentiate Eqs. (5) and (6) with respect to x and z_1, respectively:

$$\frac{\partial P_{r\theta}}{\partial x} = \frac{\partial \tau}{\partial x} \cdot \frac{\left(\frac{\partial v}{\partial x}\right)}{\left(\frac{\partial v}{\partial n}\right)} + \tau \cdot \frac{\left[\frac{\partial^2 v}{\partial x^2} \cdot \left(\frac{\partial v}{\partial z_1}\right)^2 - \frac{\partial v}{\partial x} \cdot \frac{\partial v}{\partial z_1} \cdot \frac{\partial^2 v}{\partial x \partial z_1}\right]}{\left(\frac{\partial v}{\partial n}\right)^3},$$

$$\frac{\partial P_{z\theta}}{\partial z_1} = \frac{\partial \tau}{\partial z_1} \cdot \frac{\left(\frac{\partial v}{\partial z_1}\right)}{\left(\frac{\partial v}{\partial n}\right)} + \tau \frac{\left[\frac{\partial^2 v}{\partial z_1^2} \left(\frac{\partial v}{\partial x}\right)^2 - \frac{\partial v}{\partial x} \cdot \frac{\partial v}{\partial z_1} \cdot \frac{\partial^2 v}{\partial x \partial z_1}\right]}{\left(\frac{\partial v}{\partial n}\right)^3}.$$

Let us suppose that

$$f = (x, z_1) = D \tag{7}$$

is the geometrical locus of the points at which the function $\nu(x, z_1)$ takes the value C. Essentially the curve (7) should be such that grad f is always directed in the direction of the external normal to this curve. Hence the sign of the normal derivative $\partial \nu/\partial n$ always coincides with the sign of the derivative $\partial \nu/\partial f$.

Remembering that $\partial P r_{r\theta}/\partial r = \partial P_{r\theta}/\partial x$, we may rewrite Eq. (4) in the following way:

$$\frac{i + 2\tau\cos(nx)}{r} + \frac{\partial \tau}{\partial x} \cdot \frac{\left(\frac{\partial v}{\partial x}\right)}{\left(\frac{\partial v}{\partial n}\right)} + \frac{\partial \tau}{\partial z_1} \cdot \frac{\left(\frac{\partial v}{\partial z_1}\right)}{\left(\frac{\partial v}{\partial n}\right)} + \frac{\tau\left[\frac{\partial^2 v}{\partial x^2}\left(\frac{\partial v}{\partial z_1}\right)^2 - 2\frac{\partial v}{\partial x} \cdot \frac{\partial v}{\partial z_1} \cdot \frac{\partial^2 v}{\partial x \partial z_1} + \frac{\partial^2 v}{\partial z_1^2}\left(\frac{\partial v}{\partial x}\right)^2\right]}{\left(\frac{\partial v}{\partial n}\right)^3} = 0. \tag{8}$$

Let us transform the expression

$$\frac{\frac{\partial^2 v}{\partial x^2}\left(\frac{\partial v}{\partial z_1}\right)^2 - 2\frac{\partial v}{\partial x} \cdot \frac{\partial v}{\partial z_1} \cdot \frac{\partial^2 v}{\partial x \partial z_1} + \frac{\partial^2 v}{\partial z_1^2}\left(\frac{\partial v}{\partial x}\right)^2}{\left(\frac{\partial v}{\partial n}\right)^3} = \frac{\left(\frac{\partial v}{\partial f}\right)^3\left[\frac{\partial^2 f}{\partial x^2}\left(\frac{\partial f}{\partial z_1}\right)^2 - 2\frac{\partial f}{\partial x} \cdot \frac{\partial f}{\partial z_1} \cdot \frac{\partial^2 f}{\partial x \partial z_1} + \frac{\partial^2 f}{\partial z_1^2}\left(\frac{\partial f}{\partial x}\right)^2\right]}{\pm\left|\frac{\partial v}{\partial f}\right|^3\left[\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial z_1}\right)^2\right]^{3/2}}.$$

The sign in the denominator is chosen so that it may coincide with the sign of the normal derivative $\partial \nu/\partial n$. Thus we have

$$\frac{\left(\frac{\partial v}{\partial f}\right)^3}{\pm\left|\frac{\partial v}{\partial f}\right|^3} = 1.$$

Remembering further that

$$\frac{\frac{\partial^2 f}{\partial x^2}\left(\frac{\partial f}{\partial z_1}\right)^2 - 2\frac{\partial f}{\partial x} \cdot \frac{\partial f}{\partial z_1} \cdot \frac{\partial^2 f}{\partial x \partial z_1} + \frac{\partial^2 f}{\partial z_1^2}\left(\frac{\partial f}{\partial x}\right)^2}{\left[\sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial z_1}\right)^2}\right]^3} = -\frac{1}{\rho},$$

where ρ is the radius of curvature of the isoline (7), Eq. (8) may be rewritten

$$\frac{i + 2\tau\cos(nx)}{r} + \frac{\partial \tau}{\partial n} - \frac{\tau}{\rho} = 0. \tag{9}$$

In its structure this equation is analogous to that obtained earlier by the author for the case of axial motion in a cylinder of noncircular section. By using (9) in an inverse manner we may derive an approximate solution to the problem of the motion of a viscoplastic system in an elbow pipe of noncircular cross section.

However, when the contour of the pipe cross section is given analytically, it is more convenient to use the direct method, since in this case the law governing the distribution of the tangential stresses in the solid wall may usually be completely determined.

Let us use Eq. (8) for solving the problem of motion in an elbow pipe of circular cross section by the direct method. In the present case all the normal derivatives are derivatives along the direction of the internal normal to the isoline $f(x, z_1) = D$.

Considering the inner region very close to the solid wall, we express the particular solution of Eq. (8) in the form

$$\tau_1 = \frac{i(R^2 + Rr - 2r^2)}{6r^2(R-r)} \sqrt{z_1^2 + (r-R)^2}.$$

We express the general solution of Eq. (8) in the form

$$\tau = \tau_1 + \tau_2,$$

where

$$\tau_2 = \tau_{2b} \exp \int\limits_0^{\Delta n} \left[\frac{1}{\rho} + \frac{2\cos(nx)}{r} \right] dn;$$

τ_{2b} is the value of τ_2 at the solid wall (boundary).

Let us expand the function $\nu(n)$ in powers of Δn in the neighborhood of a point lying on the contour of the pipe cross section.

If we start from the principle that the system is attached to the solid wall, the contour of the pipe cross section is the zero isoline of the function $\nu(x, z_1)$. At all boundary points of the core of the cross section this function should have a constant value, and at these points we should also have $\partial\nu/\partial n = 0$.

Using ν_0 to denote the value of the function $\nu(x, z_1)$ at the boundary of the core of the cross section we obtain

$$\nu_0 = \left(\frac{\partial \nu}{\partial n}\right)_b \Delta n + \left(\frac{\partial^2 \nu}{\partial n^2}\right)_b \frac{\Delta n^2}{2} + \ldots \tag{10}$$

The tangential stress at the solid wall is

$$\tau_b = \tau_{1b} + \tau_{2b} = \tau_0 + (R + r_0 \cos\varphi)\, \eta \left(\frac{\partial \nu}{\partial n}\right)_b,$$

where r_0 is the radius of the pipe cross section. From this equation we obtain

$$\left(\frac{\partial \nu}{\partial n}\right)_b = \frac{\tau_{1b} + \tau_{2b} - \tau_0}{\eta (R + r_0 \cos\varphi)}. \tag{11}$$

Differentiating the equation

$$\tau = \tau_0 + \eta r \frac{\partial \nu}{\partial n},$$

we obtain

$$\frac{\partial \tau}{\partial n} = \eta \left(\frac{\partial r}{\partial n} \cdot \frac{\partial \nu}{\partial n} + r \frac{\partial^2 \nu}{\partial n^2} \right).$$

For points close to the solid wall, and subject to the condition that n is the internal normal to the isoline $\nu(x, z_1) = C$, we have

$$\frac{\partial \tau}{\partial n} = \eta \left[\frac{R-r}{\sqrt{z_1^2 + (r-R)^2}} \cdot \frac{\partial \nu}{\partial n} + r \frac{\partial^2 \nu}{\partial n^2} \right]. \tag{12}$$

On the other hand, from Eq. (9) we have

$$\frac{\partial\tau}{\partial n}=\frac{\tau}{|\rho|}-\frac{[i-2\tau\cos(nx)]}{r}. \qquad (13)$$

Equating the right-hand sides of (12) and (13), at the solid wall we obtain

$$\left(\frac{\partial^2 v}{\partial n^2}\right)_b=\frac{(\tau_{1b}+\tau_{2b})R-r_0(i+\tau_0\cos\varphi)}{r_0(R+r_0\cos\varphi)^2\eta}. \qquad (14)$$

If in Eq. (10) we take only the first two terms of the expansion on the right-hand side, we have

$$\Delta n=\frac{-\left(\frac{\partial v}{\partial n}\right)_b\pm\sqrt{\left(\frac{\partial v}{\partial n}\right)_b^2+2v_0\left(\frac{\partial^2 v}{\partial n^2}\right)_b}}{\left(\frac{\partial^2 v}{\partial n^2}\right)_b}. \qquad (15)$$

It follows from the condition that, at any point of the boundary, the tangential stress should equal the limiting shear stress that

$$\left(\frac{\partial v}{\partial n}\right)_0=\left(\frac{\partial v}{\partial n}\right)_b+\left(\frac{\partial^2 v}{\partial n}\right)_b\Delta n=0.$$

In order to satisfy this condition the expression under the root in Eq. (15) must be equated to zero:

$$\left(\frac{\partial v}{\partial n}\right)_b^2+2v_0\left(\frac{\partial^2 v}{\partial n^2}\right)_b=0.$$

Substituting the values of the derivatives from (11) and (14) into this we have

$$r_0(\tau_{1b}^2+\tau_{2b}^2+\tau_0+2\tau_{1b}\tau_{2b}-2\tau_{1b}\tau_0-2\tau_{2b}\tau_0)+2v_0\eta\left[\tau_{1b}R+\tau_{2b}R-r_0(i+\tau_0\cos\varphi)\right]=0.$$

From this equation we find

$$\tau_{2b}=\tau_0-\tau_{1b}-v_0\eta\beta\pm\sqrt{2v_0\eta[i+\tau_0(\cos\varphi-\beta)]+v_0^2\eta^2\beta^2},$$

where

$$\beta=R/r_0.$$

The tangential stress at the solid wall

$$\tau_b=\tau_{1b}+\tau_{2b}=\tau_0-v\eta\beta\pm\sqrt{2v_0\eta[i+\tau_0(\cos\varphi-\beta)]+v_0^2\eta^2\beta^2}. \qquad (16)$$

From the physical point of view the tangential stress τ_b must have a positive sign; furthermore, in modulus it should be greater than the limiting shear stress at all points. In order to satisfy this condition we must take the plus sign in front of the root in the last equation.

In Eq. (16) the quantity $\nu_0^2\eta^2\beta^2$ under the root is very small compared with the other terms and may be neglected.

From the condition of equilibrium of an elementary volume of dispersed mass in the elbow pipe we have

$$\int_0^{2\pi}\tau_b(R+r_0\cos\varphi)\,d\varphi=i\pi r_0.$$

Putting the value of τ_b into this we obtain

$$(\tau_0-v_0\eta\beta)\,2\pi R\pm\sqrt{2v_0\eta}\,(I_1R+I_2r_0)=i\pi r_0, \qquad (17)$$

where

$$I_1=\int_0^{2\pi}\sqrt{i+\tau_0(\cos\varphi-\beta)}\,d\varphi;$$

$$I_2=\int_0^{2\pi}\cos\varphi\sqrt{i+\tau_0(\cos\varphi-\beta)}\,d\varphi.$$

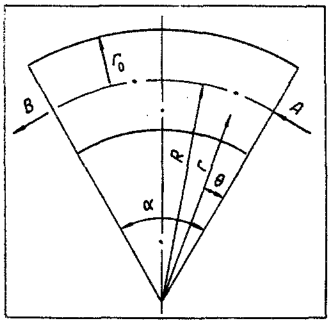

Fig. 1

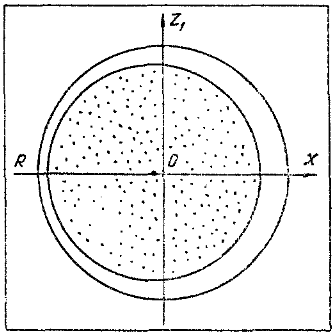

Fig. 2

Fig. 1. Part of the pipe in which the flow occurs.

Fig. 2. Disposition of the core of the flow in the circular cross section.

In solving Eq. (16) with respect to ν_0 we must take the plus sign in front of the root, since it is clear from the physical meaning that with increasing i the modulus of ν_0 should increase.

According to (16), for $\nu_0 = 0$, $\tau_b = \tau_0$. Substituting for τ_b in (17) by inserting the numerical value of τ_0 we find the minimum value of i required for the onset of viscoplastic flow:

$$i_{\min} = 2\tau_0\beta.$$

Corresponding to this value of $i_{\min}$ we have the minimum pressure at the inlet ($P_1 = 0$):

$$P_{1\min} = i_{\min}\,\alpha.$$

Equation (11) takes the following form after substituting the values of τ_{1b} and τ_{2b}

$$\left(\frac{\partial v}{\partial n}\right)_b = \frac{-v_0\eta\beta + \sqrt{2v_0\eta\,[i + \tau_0(\cos\varphi - \beta)]}}{\eta(R + r_0\cos\varphi)}.$$

Putting the same values into (14) we obtain

$$\left(\frac{\partial^2 v}{\partial n^2}\right)_b = \frac{\{\tau_0 - v_0\eta\beta + \sqrt{2v_0\eta\,[i + \tau_0(\cos\varphi - \beta)]}\}\,R - r_0(i + \tau_0\cos\varphi)}{\eta r_0(R + r_0\cos\varphi)^2}.$$

The distance from the solid wall to the boundary of the core of the cross section will be

$$\Delta n = -\frac{\left(\frac{\partial v}{\partial n}\right)_b}{\left(\frac{\partial^2 v}{\partial n^2}\right)_b}$$

or

$$\Delta n = \frac{\{v_0\eta\beta - \sqrt{2v_0\eta\,[i + \tau_0(\cos\varphi - \beta)]}\}\,r_0(R + r_0\cos\varphi)}{\{\tau_0 - v\eta\beta + \sqrt{2v_0\eta\,[i + \tau_0(\cos\varphi - \beta)]}\}\,R - r_0(i + \tau_0\cos\varphi)}.$$

The core of the cross section derived from this equation takes the form illustrated in Fig. 2.

NOTATION

R	is the radius of the axis of the elbow pipe;
r_0	is the radius of the pipe cross section;
P	is the hydrostatic pressure;
ρ_0	is the density;
v_θ, v_r, v_z	are the velocity components;
τ_0	is the limiting shear stress;
η	is the plastic viscosity;
τ	is the principal tangential stress;

τ_b is the tangential stress at the solid wall;

β is the ratio R/r_0;

α is the central angle of the arc AB;

φ is the angle between the normal and the ox axis;

Δn is the distance from the solid wall to the boundary of the core of the cross section.

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